

A Strange Partition Theorem Related to the Second Atkin-Garvan Moment

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Abstract

This paper contains results on a strange smallest parts function related to the second Atkin-Garvan moment. Some new identities are discovered in relation to Andrews' *spt* function as well as one of Borweins' two-dimensional theta functions.

1 Introduction

The generating function for the sum of smallest parts among partitions of n into an odd number of distinct parts minus the sum of the smallest parts among the partitions of n into an even number of distinct parts is

$$\sum_{n \geq 0} (1 - (q^{n+1})_{\infty}),$$

where we use standard notation [8] $(a)_n = (a; q)_n := \prod_{0 \leq k \leq n-1} (1 - aq^k)$, $q \in \mathbb{C}$. In fact, it is known that (see [2])

$$\sum_{n \geq 0} (1 - (q^{n+1})_{\infty}) = \sum_{n \geq 1} \sigma_0(n) q^n, \quad (1)$$

where $\sigma_k(n) = \sum_{d|n} d^k$, $n \in \mathbb{N}$, and $\sigma_k(n) = 0$ if $n \notin \mathbb{N}$. (See [11] for similar identities.)

Andrews' [2] discovered the generating function for *spt*(n), the total number of appearances of the smallest part in unrestricted partitions of n , and then related it to the second Atkin-Garvan Moment. Namely, he proved [2]

$$\sum_{n \geq 1} \text{spt}(n) q^n = \sum_{n \geq 1} \frac{q^n}{(1 - q^n)(q^n)_{\infty}} = \sum_{n \geq 1} np(n) q^n - \frac{1}{2} \sum_{n \geq 1} N_2(n) q^n, \quad (2)$$

where $p(n)$ is the number of unrestricted partitions of n , $N_2(n) = \sum_{m \in \mathbb{Z}} m^2 N(m, n)$ is the second Atkin-Garvan moment, and $N(m, n)$ is the number of partitions of n with rank m . The “rank” of a partition is the largest part minus the number of parts. The implication of relating

$spt(n)$ to the right side of equation (2) is that $spt(n)$ satisfies some interesting Ramanujan-type congruences [1, 2].

The purpose of this paper is to offer a strange result similar to (2), and more than the contents of my preprint referenced in Garvan's paper [8].

Theorem 1. *We have,*

$$\begin{aligned} \sum_{n \geq 1} \frac{q^n}{(1-q^n)(1-q^n)(1-q^{n+1}) \cdots (1-q^{2n-1})(q^3; q^3)_\infty} \\ = \frac{1}{(q^3; q^3)_\infty} \sum_{n \geq 1} \frac{nq^n}{1-q^n} - \frac{1}{2} \sum_{n \geq 1} N_2(n)q^{3n}. \end{aligned} \quad (3)$$

To prove this theorem, we require the machinery of Bailey pairs and Bailey's lemma. Recall [4] that we define a pair of sequences (α_n, β_n) to be a Bailey pair with respect to a if

$$\beta_n = \sum_{r=0}^n \frac{\alpha_r}{(aq; q)_{n+r}(q; q)_{n-r}}. \quad (4)$$

The following is Bailey's lemma, which utilizes this definition of (α_n, β_n) to produce new q -series identities.

Lemma 1.2 *If (α_n, β_n) form a Bailey pair with respect to a then*

$$\sum_{n \geq 0} (z)_n (y)_n (aq/zy)^n \beta_n = \frac{(aq/z)_\infty (aq/y)_\infty}{(aq)_\infty (aq/zy)_\infty} \sum_{n \geq 0} \frac{(z)_n (y)_n (aq/zy)^n \alpha_n}{(aq/z)_n (aq/y)_n}. \quad (5)$$

From Slater's list, we have the following Bailey pair (α_n, β_n) relative to $a = 1$ [12, J(1)]

$$\alpha_{3n \pm 1} = 0, \quad (6)$$

$$\alpha_{3n} = (-1)^n q^{3n(3n-1)/2} (1 + q^{3n}), \quad (7)$$

$$\beta_n = \frac{(q^3; q^3)_{n-1}}{(q)_n (q)_{2n-1}}. \quad (8)$$

Differentiating (5) with respect to z and setting $z = 1$, and then doing the same for y yields the identity (after setting $a = 1$),

$$\sum_{n \geq 1} (q; q)_{n-1}^2 \beta_n q^n = \alpha_0 \sum_{n \geq 1} \frac{nq^n}{1-q^n} + \sum_{n \geq 1} \frac{\alpha_n q^n}{(1-q^n)^2}. \quad (9)$$

Inserting the Bailey pair (6)-(8) into equation (9) and then multiplying through by $(q^3; q^3)_\infty^{-1}$ gives us Theorem 1. To see that

$$-\frac{1}{2} \sum_{n \geq 1} N_2(n) q^{3n} = \sum_{n \geq 1} \frac{\alpha_n q^n}{(1-q^n)^2},$$

note [2, eq.(3.4)]

$$-\frac{1}{2} \sum_{n \geq 1} N_2(n) q^n = \sum_{n \geq 1} \frac{(-1)^n q^{n(3n+1)/2} (1 + q^n)}{(1 - q^n)^2}.$$

The generating function on the left hand side of (3) is $spt_{2,3}(n)$, the total number of appearances of the smallest part in each integer partition of n , where parts are $<$ twice the smallest or multiples of three \geq thrice the smallest.

2 A Relation to the Borwein theta function $a(q)$

One of the theta functions introduced by the Borwein's [5, Chapter 4], [6] is the function

$$a(q) := \sum_{n, m \in \mathbb{Z}} q^{n^2 + nm + m^2}. \quad (10)$$

The function $a(q)$ has been studied in considerable detail, and has several direct relations to Jacobi's theta functions (see [6]). The Lambert series expansion for $a(q)$ is due to Lorenz, and can be found in [6, eq.(2.21)]

$$a(q) = 1 + 6 \sum_{n \geq 1} \left(\frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right).$$

In [3, pg. 460, Entry 3(i)], we find

$$\sum_{n \geq 1} \frac{nq^n}{1 - q^n} - 3 \sum_{n \geq 1} \frac{nq^{3n}}{1 - q^{3n}} = \frac{a^2(q) - 1}{12}. \quad (11)$$

We are now ready to prove the following result.

Theorem 2. *If $n \equiv \pm 1 \pmod{3}$, and*

$$\sum_{m \geq 1} \xi(m) q^m := \frac{1}{(q^3; q^3)_\infty} \left(\frac{a^2(q) - 1}{12} \right),$$

then $spt_{2,3}(n) = \xi(n)$. Further, if $n \equiv 0 \pmod{3}$, then

$$spt_{2,3}(n) = 3spt(n/3) + \xi(n) + N_2(n/3).$$

The key identity to prove this result is,

$$\begin{aligned} & \sum_{n \geq 1} spt_{2,3}(n) q^n - 3 \sum_{n \geq 1} spt(n) q^{3n} \\ &= \frac{1}{(q^3; q^3)_\infty} \left(\frac{a^2(q) - 1}{12} \right) - \frac{2}{(q^3; q^3)_\infty} \sum_{n \geq 1} \frac{(-1)^n q^{3n(3n+1)/2} (1 + q^{3n})}{(1 - q^{3n})^2}. \end{aligned} \quad (12)$$

Equation (12) is three multiplied by equation (2) (with q replaced by q^3) subtracted from Theorem 2. Note the last series on the right side of (12) is $\sum_{n \geq 1} N_2(n)q^{3n}$ [2. eq.(3.4)]. The first part of Theorem 2 is obtained from taking the coefficient of $q^{3n \pm 1}$ of equation (12), and noting that the generating functions over q^{3n} get omitted when doing this. The second part follows from taking the coefficient of q^{3n} on both sides.

3 Another look at Theorem 2

In a paper by Huard, Ou, Spearman, and Williams [10] on convolution sums involving divisor functions, we find the following nice theorem:

[10, Theorem 13] *The number of representations of a positive integer n by the quaternary form $x^2 + xy + y^2 + u^2 + uv + v^2$ is $12\sigma(n) - 36\sigma(n/3)$.*

This theorem is attributed in [10] to G. A. Lomadze. It is not difficult to see that [10, Theorem 13] is essentially equivalent to equation (11) upon taking the coefficient of q^n . Naturally, from this observation we can restate Theorem 2 using the cardinality of a quaternary form.

Following their notation [10], put

$$R(k) = \mathbf{card}\{(x, y, u, v) \in \mathbb{Z}^4 \mid k = x^2 + xy + y^2 + u^2 + uv + v^2\}.$$

Using the above result we can now prove the following.

Theorem 3. *Let $p_3(n)$ be the number of partitions of n with parts congruent to 0 (mod 3).*

Put

$$P_3(n) = \sum_k R(k)p_3(n - k),$$

then $spt_{2,3}(3n) \equiv \frac{1}{12}P_3(3n) - \frac{1}{2}N_2(n) \pmod{3}$. Further, if $n \equiv \pm 1 \pmod{3}$ then $spt_{2,3}(n) = \frac{1}{12}P_3(n)$.

Proof. Using [10, Theorem 13], it can be seen that

$$\frac{1}{(q^3; q^3)_\infty} \sum_{n \geq 1} \frac{nq^n}{1 - q^n} \quad (13)$$

$$= \frac{1}{(q^3; q^3)_\infty} \sum_{n \geq 1} \sigma(n)q^n \quad (14)$$

$$= \frac{1}{(q^3; q^3)_\infty} \sum_{n \geq 1} \left(\frac{1}{12} R(n) + 3\sigma(n/3) \right) q^n \quad (15)$$

$$= \frac{1}{(q^3; q^3)_\infty} \left(\frac{1}{12} \sum_{n \geq 1} R(n)q^n + 3 \sum_{n \geq 1} \frac{nq^{3n}}{1 - q^{3n}} \right) \quad (16)$$

$$= \frac{1}{12} \sum_{n \geq 1} P_3(n)q^n + 3 \sum_{n \geq 1} np(n)q^{3n}, \quad (17)$$

since $\sigma(n/3) = 0$ unless $n \equiv 0 \pmod{3}$ by definition. Line (17) follows from Euer's well-known identity $np(n) = \sum_{k \leq n} p(k)\sigma(n-k)$. This observation coupled with equation (3), and equating coefficients of q^{3n} and then $q^{3n \pm 1}$ gives the result. \square

We note that one might also write

$$P_3(3n) = \sum_k R(3k)p(n-k),$$

upon noting that the coefficient of q^{3n} of $\sum_{n \geq 0} P_3(n)q^n$ is the coefficient of q^{3n} in

$$\sum_{m, k \geq 0} p(m)R(3k)q^{3(m+k)}.$$

4 A Relation to Andrews' spt function

In this section we offer a nice consequence of Theorem 1 and some concluding remarks.

Theorem 4. *We have, $spt_{2,3}(3n) \equiv spt(n) \pmod{3}$.*

Proof. By Theorem 1, we have

$$\sum_{n \geq 1} spt_{2,3}(n)q^n = \frac{1}{(q^3; q^3)_\infty} \sum_{n \geq 1} \sigma(n)q^n - \frac{1}{2} \sum_{n \geq 1} N_2(n)q^{3n}. \quad (18)$$

Note that $\sigma(3n) = 4\sigma(n) - 3\sigma(n/3)$, and hence $\sigma(3n) \equiv 4\sigma(n) \pmod{3}$, and therefore $\sigma(3n) \equiv \sigma(n) \pmod{3}$ (by the triviality $a \equiv 4b \pmod{3}$ iff $a \equiv b \pmod{3}$). With this in mind, we see that the coefficient of q^{3n} in (18) is

$$spt_{2,3}(3n) = \sum_k p(k)\sigma(3(n-k)) - \frac{1}{2} N_2(n). \quad (19)$$

This follows from the observation that the coefficient of q^{3n} in

$$\sum_{k \geq 1} p(k) q^{3k} \sum_{m \geq 1} \sigma(m) q^m,$$

is the coefficient of q^{3n} in

$$\sum_{m, k \geq 1} p(k) \sigma(3m) q^{3(m+k)}.$$

Now using $\sigma(3n) \equiv \sigma(n) \pmod{3}$ with equation (19), we find

$$spt_{2,3}(3n) \tag{20}$$

$$\equiv \sum_k p(k) \sigma(n-k) - \frac{1}{2} N_2(n) \pmod{3} \tag{21}$$

$$\equiv np(n) - \frac{1}{2} N_2(n) \pmod{3} \tag{22}$$

$$\equiv spt(n) \pmod{3}. \tag{23}$$

□

For a complete multiplicative theory of Andrews' spt function modulo 3 see [7]. It would be nice to see a similar theory built for $spt_{2,3}(n)$ using Theorem 4.

We note it is natural to consider re-studying Theorem 2 (or $\xi(n)$) further by considering further identities for the q -series

$$\sum_{n, m, i, j \in \mathbb{Z}} q^{n^2 + m^2 + i^2 + j^2 + ij + nm} = a^2(q).$$

In particular, from [6, pg.37, eq.(2.1)] and [6, pg.37, Proposition 2.2] we have the known identity

$$a(q) = 9q \frac{(q^9; q^9)_\infty^3}{(q^3; q^3)_\infty} + \frac{(q; q)_\infty^3}{(q^3; q^3)_\infty}. \tag{24}$$

Hence,

$$\begin{aligned} & \sum_{n \geq 1} spt_{2,3}(n) q^n - 3 \sum_{n \geq 1} spt(n) q^{3n} \\ &= \frac{1}{(q^3; q^3)_\infty} \left(\frac{27}{4} q^2 \frac{(q^9; q^9)_\infty^6}{(q^3; q^3)_\infty^2} + \frac{3q}{2} \frac{(q; q)_\infty^3 (q^9; q^9)_\infty^3}{(q^3; q^3)_\infty^2} + \frac{1}{12} \frac{(q; q)_\infty^6}{(q^3; q^3)_\infty^2} - \frac{1}{12} \right) + \sum_{n \geq 1} N_2(n) q^{3n} \end{aligned} \tag{25}$$

$$\equiv \frac{1}{12} \frac{(q; q)_\infty^6}{(q^3; q^3)_\infty^3} - \frac{1}{12 (q^3; q^3)_\infty} + \sum_{n \geq 1} N_2(n) q^{3n} \pmod{3} \tag{26}$$

Therefore,

$$\sum_{n \geq 1} spt_{2,3}(n)q^n \equiv \frac{1}{12} \frac{(q; q)_\infty^6}{(q^3; q^3)_\infty^3} - \frac{1}{12(q^3; q^3)_\infty} + \sum_{n \geq 1} N_2(n)q^{3n} \pmod{3}. \quad (27)$$

On the other hand, we may write

$$\frac{1}{12} \frac{(q; q)_\infty^6}{(q^3; q^3)_\infty^3} = \frac{1}{12} \left(\sum_{n \geq 0} (-1)^n (2n+1) q^{n(n+1)/2} \right)^2 \left(\sum_{n \geq 0} p(n) q^{3n} \right)^3. \quad (28)$$

Since we may write

$$\sum_{n \geq 0} f_n q^{3n} = \left(\sum_{n \geq 0} p(n) q^{3n} \right)^3,$$

where f_n is a convolution sum involving $p(n)$, we are concerned primarily when the sum of two triangular numbers is $\equiv 2 \pmod{3}$. If we write $T_i = i(i+1)/2$, $i, j \in \mathbb{N}$, then we have that $T_i + T_j \equiv 2 \pmod{3}$ only when both $i \equiv 1 \pmod{3}$, $j \equiv 1 \pmod{3}$. When this occurs we see that $(2i+1)(2j+1)$ is of the form $9(2i'+1)(2j'+1)$, $i', j' \in \mathbb{N}$. Therefore, taking the coefficient of q^{3n+2} in (27) now gives us the following result.

Theorem 5. $spt_{2,3}(3n+2) \equiv 0 \pmod{3}$.

To see some examples numerically (recall the restriction on parts that parts are $<$ twice the smallest or multiples of three \geq thrice the smallest), we have the following examples:

Example 1: $spt_{2,3}(5) = 9 \equiv 0 \pmod{3}$. Since we are to count the number of appearances of the smallest parts in the partitions (5), (3, 2), (3, 1, 1), (1, 1, 1, 1, 1).

Example 2: $spt_{2,3}(8) = 27 \equiv 0 \pmod{3}$. Since we are to count the number of appearances of the smallest parts in the partitions (8), (6, 2), (6, 1, 1), (5, 3), (4, 4), (3, 3, 2), (3, 3, 1, 1), (3, 1, 1, 1, 1, 1), (2, 2, 2, 2), (1, 1, 1, 1, 1, 1, 1, 1).

References

- [1] G. E. Andrews, *The Theory of Partitions*, The Encyclopedia of Mathematics and its Applications, Vol. 2, Addison-Wesley, Reading (1976).
- [2] G. E. Andrews, *The number of smallest parts in the partitions of n* , J. Reine Angew. Math. 624 (2008), 133–142.
- [3] B. C. Berndt, *Ramanujans Notebook*, Part III, Springer Verlag, New York, 1991.

- [4] W. N. Bailey, *Identities of the Rogers–Ramanujan type*, Proc. London Math. Soc. (2), 50 (1949), 1–10.
- [5] J.M. Borwein and P.B. Borwein, *Pi and the AGM – A Study in Analytic Number Theory and Computational Complexity*, Wiley, N.Y., 1987.
- [6] J.M. Borwein and P.B. Borwein, F. G. Garvan, *Some Cubic Modular Identities of Ramanujan*, Transactions of the American Mathematical Society, Vol. 343, No.1 1994, pp. 35–47.
- [7] A. Folsom and K. Ono, *The spt-function of Andrews*, Proc. Natl. Acad. Sci. USA 105 (2008), 20152–20156.
- [8] F. Garvan, *Higher Order spt-Functions*, Adv. in Math. 228 (2011), 241–265.
- [9] G. Gasper, M. Rahman, *Basic hypergeometric series*, Cambridge Univ. Press, Cambridge, 1990.
- [10] James G. Huard, Zhiming M. Ou, Blair K. Spearman, and Kenneth S. Williams, *Elementary evaluation of certain convolution sums involving divisor functions*, Number Theory for the Millennium II, edited by M. A. Bennett, B. C. Berndt, N. Boston, H. G. Diamond, A. J. Hildebrand, and W. Philipp, A. K. Peters, Natick, Massachusetts, 2002, pp. 229–274.
- [11] A. E. Patkowski, *Divisors, partitions and some new q-series identities*, Colloq. Math. 117 (2009), 289–294.
- [12] L. J. Slater, *Further identities of the Rogers–Ramanujan type*, Proc. London Math. Soc. (2), 54:147–167, 1952.

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